

*Extensions of the Poincaré Group**Ignatios Antoniadis^{1†}, Lars Brink² and George Savvidy¹³*¹ *Department of Physics, CERN Theory Division CH-1211 Geneva 23, Switzerland*² *Department of Fundamental Physics,
Chalmers University of Technology, S-412 96 Göteborg, Sweden*³ *Demokritos National Research Center, Ag. Paraskevi, Athens, Greece***Abstract**

We construct an extension of the Poincaré group which involves a mixture of internal and space-time supersymmetries. The resulting group is an extension of the superPoincaré group with infinitely many generators which carry internal and space-time indices. It is a closed algebra since all Jacobi identities are satisfied and it has therefore explicit matrix representations. We investigate the massless case and construct the irreducible representations of the extended symmetry. They are divided into two sets, longitudinal and transversal representations. The transversal representations involve an infinite series of integer and half-integer helicities. Finally we suggest an extension of the conformal group along the same line.

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1 Introduction

In an attempt to discuss higher spin gauge fields in a new setting, a generalization of the Poincaré algebra has been suggested [1]. In it the Poincaré generators are enlarged by infinitely many new bosonic generators which carry internal and space-time indices. In this article we shall construct a supersymmetric extension of this algebra. The resulting algebra contains the ordinary superPoincaré generators together with infinitely many bosonic generators which form a current algebra between themselves. It is a closed algebra since all Jacobi identities are satisfied and it can hence have explicit matrix representations.

Let us first introduce the infinite set of translationally invariant generators which carry internal and space-time indices:

$$L_a^{\lambda_1 \dots \lambda_s}, \quad s = 0, 1, 2, \dots \quad (1)$$

where L_a ($s = 0$) are the generators of the internal Lie algebra L_G and the generators $L_a^{\lambda_1 \dots \lambda_s}$ are totally symmetric with respect to the indices $\lambda_1 \dots \lambda_s$. These generators carry space-time and internal indices and transform under the operations of both groups. In a sense these generators remind us of gauge fields having both Lorentz and internal indices and, as we shall see, there are some properties inherent of gauge fields in them. The current algebra of these generators is defined as follows [1]:

$$[L_a^{\lambda_1 \dots \lambda_n}, L_b^{\lambda_{n+1} \dots \lambda_s}] = i f_{abc} L_c^{\lambda_1 \dots \lambda_s}, \quad s = 0, 1, 2, \dots \quad (2)$$

where at the basic level ($s=0$) it contains the internal algebra L_G with commutators $[L_a, L_b] = i f_{abc} L_c$. The current algebra (2) is not yet completely defined because it does not specify how the new generators $L_a^{\lambda_1 \dots \lambda_s}$ transform under space-time transformations. Assuming the generators $L_a^{\lambda_1 \dots \lambda_s}$ be translationally invariant tensors of rank s , the following extension of the Poincaré algebra was suggested [1]:

$$[P^\mu, P^\nu] = 0, \quad (3)$$

$$[M^{\mu\nu}, P^\lambda] = i(\eta^{\lambda\nu} P^\mu - \eta^{\lambda\mu} P^\nu),$$

$$[M^{\mu\nu}, M^{\lambda\rho}] = i(\eta^{\mu\rho} M^{\nu\lambda} - \eta^{\mu\lambda} M^{\nu\rho} + \eta^{\nu\lambda} M^{\mu\rho} - \eta^{\nu\rho} M^{\mu\lambda}),$$

$$[P^\mu, L_a^{\lambda_1 \dots \lambda_s}] = 0, \quad (4)$$

$$[M^{\mu\nu}, L_a^{\lambda_1 \dots \lambda_s}] = i(\eta^{\lambda_1\nu} L_a^{\mu\lambda_2 \dots \lambda_s} - \eta^{\lambda_1\mu} L_a^{\nu\lambda_2 \dots \lambda_s} + \dots + \eta^{\lambda_s\nu} L_a^{\lambda_1 \dots \lambda_{s-1}\mu} - \eta^{\lambda_s\mu} L_a^{\lambda_1 \dots \lambda_{s-1}\nu}),$$

$$[L_a^{\lambda_1 \dots \lambda_n}, L_b^{\lambda_{n+1} \dots \lambda_s}] = i f_{abc} L_c^{\lambda_1 \dots \lambda_s} \quad (s = 0, 1, 2, \dots). \quad (5)$$

The first three commutators define the Poincaré algebra as its subalgebra. The next two commutators tell us that the generators $L_a^{\lambda_1 \dots \lambda_s}$ are translationally invariant tensors of rank s and the last commutator defines the current subalgebra (2). One can check that all Jacobi identities are satisfied and we have an example of fully consistent algebra, which is called an *extended Poincaré algebra* $L_G(\mathcal{P})$ associated with a compact Lie group G . Thus the algebra $L_G(\mathcal{P})$ incorporates the Poincaré algebra and an internal algebra L_G in a nontrivial way, which is different from the direct product. The generators $L_a^{\lambda_1 \dots \lambda_s}$ have a nonzero commutation relation with $M^{\mu\nu}$ and therefore carry higher spins.

2 Supersymmetric Extension of the $L_G(\mathcal{P})$ Algebra

We are interested in constructing further extensions of the $L_G(\mathcal{P})$ algebra which should include anti-commuting generators. *A priory* it is not obvious that such an extension can be constructed. With this intention in mind, let us compare the above extension of the Poincaré algebra with the (extended) super-Poincaré algebra which is defined as follows [3, 4, 5, 6, 7, 8]:

$$[P^\mu, P^\nu] = 0, \quad (6)$$

$$\begin{aligned} [M^{\mu\nu}, P^\lambda] &= i(\eta^{\lambda\nu} P^\mu - \eta^{\lambda\mu} P^\nu), \\ [M^{\mu\nu}, M^{\lambda\rho}] &= i(\eta^{\mu\rho} M^{\nu\lambda} - \eta^{\mu\lambda} M^{\nu\rho} + \eta^{\nu\lambda} M^{\mu\rho} - \eta^{\nu\rho} M^{\mu\lambda}), \end{aligned}$$

$$[P^\mu, Q_\alpha^i] = 0, \quad (7)$$

$$[M^{\mu\nu}, Q_\alpha^i] = \frac{i}{2}(\gamma^{\mu\nu} Q^i)_\alpha, \quad \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$$

$$\{Q_\alpha^i, Q_\beta^j\} = -2 \delta^{ij}(\gamma^\mu C)_{\alpha\beta} P_\mu, \quad i = 1, \dots, N, \quad (8)$$

where we allowed for an R -symmetry specified by the indices i and j . Q_α^i is a Majorana spinor. This algebra also has the Poincaré algebra (3), (6) as a subalgebra. The next two commutators (4) and (7) express the fact that the extended generators Q_α^i and $L_a^{\lambda_1 \dots \lambda_s}$ are translationally invariant operators and carry a nonzero spin. The last commutators (5) and (8) are essentially different in both algebras: in super-Poincaré algebra the generators Q_α^i anti-commute with the operator P^μ , while in our case $L_a^{\lambda_1 \dots \lambda_s}$ commute with themselves to form an infinite series of commutators of the current algebra (2) which cannot be truncated. Therefore, the index s runs from zero to infinity, providing an example of an infinitely-dimensional current subalgebra [9].

Another possibility to combine the above algebras would be to consider an infinite set of spinor-tensor generators $Q_{\alpha\lambda_1 \dots \lambda_s}^i$, but this does not work. Therefore the natural suggestion is the following unification of these algebras²:

$$[P^\mu, P^\nu] = 0, \quad (9)$$

$$\begin{aligned} [M^{\mu\nu}, P^\lambda] &= i(\eta^{\lambda\nu} P^\mu - \eta^{\lambda\mu} P^\nu), \\ [M^{\mu\nu}, M^{\lambda\rho}] &= i(\eta^{\mu\rho} M^{\nu\lambda} - \eta^{\mu\lambda} M^{\nu\rho} + \eta^{\nu\lambda} M^{\mu\rho} - \eta^{\nu\rho} M^{\mu\lambda}), \end{aligned}$$

$$[P^\mu, L_a^{\lambda_1 \dots \lambda_s}] = 0, \quad (10)$$

$$\begin{aligned} [P^\mu, Q_\alpha^i] &= 0, \\ [M^{\mu\nu}, L_a^{\lambda_1 \dots \lambda_s}] &= i(\eta^{\lambda_1\nu} L_a^{\mu\lambda_2 \dots \lambda_s} - \eta^{\lambda_1\mu} L_a^{\nu\lambda_2 \dots \lambda_s} + \dots + \eta^{\lambda_s\nu} L_a^{\lambda_1 \dots \lambda_{s-1}\mu} - \eta^{\lambda_s\mu} L_a^{\lambda_1 \dots \lambda_{s-1}\nu}), \\ [M^{\mu\nu}, Q_\alpha^i] &= \frac{i}{2}(\gamma^{\mu\nu} Q^i)_\alpha, \quad \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \end{aligned}$$

$$[L_a^{\lambda_1 \dots \lambda_n}, L_b^{\lambda_{n+1} \dots \lambda_s}] = i f_{abc} L_c^{\lambda_1 \dots \lambda_s}, \quad s = 0, 1, 2, \dots$$

$$\{Q_\alpha^i, Q_\beta^j\} = -2 \delta^{ij}(\gamma^\mu C)_{\alpha\beta} P_\mu, \quad i = 1, \dots, N$$

$$[L_a^{\lambda_1 \dots \lambda_s}, Q_\alpha^i] = 0. \quad (11)$$

²Our notational conventions follow the article [2].

Here, at $s = 0$, we have the relations

$$[P^\mu, L_a] = 0, \quad [M^{\mu\nu}, L_a] = 0,$$

therefore the internal bosonic algebra L_G obeys the Coleman-Mandula theorem [10].

Let us now investigate the commutators between Q_α^i and the rest of the generators. First we have to check the Jacobi identities which contain at least one anticommuting generator. They are:

$$\begin{aligned} [[L_a^{\lambda_1 \dots \lambda_s}, P^\mu] Q_\alpha^i] + Perm. &= 0 \\ [[L_a^{\lambda_1 \dots \lambda_s}, M^{\mu\nu}] Q_\alpha^i] + Perm. &= 0 \\ [[L_a^{\lambda_1 \dots \lambda_n}, L_b^{\lambda_{n+1} \dots \lambda_s}] Q_\alpha^i] + Perm. &= 0 \end{aligned}$$

and as one can check, they are indeed identically true. The identities with two anticommuting generators have the form

$$\{ [L_a^{\lambda_1 \dots \lambda_s}, Q_\alpha^i] Q_\beta^j \} + \{ [L_a^{\lambda_1 \dots \lambda_s}, Q_\beta^j] Q_\alpha^i \} + [\{ Q_\alpha^i, Q_\beta^j \}] L_a^{\lambda_1 \dots \lambda_s} = 0$$

and they are also true. The rest of the identities are satisfied since they coincide with the identities of the known subalgebras (3)-(5) and (6)-(8).

3 General Properties of the Extended Algebra $L_G(\mathcal{SP})$

The algebra (9) is invariant with respect to the following "gauge" transformations:

$$\begin{aligned} L_a^{\lambda_1 \dots \lambda_s} &\rightarrow L_a^{\lambda_1 \dots \lambda_s} + \sum_1 P^{\lambda_1} L_a^{\lambda_2 \dots \lambda_s} + \sum_2 P^{\lambda_1} P^{\lambda_2} L_a^{\lambda_3 \dots \lambda_s} + \dots + P^{\lambda_1} \dots P^{\lambda_s} L_a \\ P^\lambda &\rightarrow P^\lambda, \\ M^{\mu\nu} &\rightarrow M^{\mu\nu}, \\ Q_\alpha^i &\rightarrow Q_\alpha^i, \end{aligned} \tag{12}$$

where the sums \sum_1, \sum_2, \dots extend over all inequivalent index permutations. It is not an internal isomorphism since it cannot be represented as conjugations by elements U of the group itself: $L \rightarrow U^{-1} L U$. The transformations contain polynomials of the commuting momenta and are reminiscent of the gauge transformations for the gauge fields. They are "off-shell" transformations because the invariant operator P^2 can have any value³. As a result, to any given representation of $L_a^{\lambda_1 \dots \lambda_s}$, $s = 1, 2, \dots$ of the extended algebra one can add the longitudinal terms, as it follows from the transformation (12). Thus all representations are defined modulo "gauge transformations" and we can identify these generators as "gauge generators".

Theorem. To any given representation of the gauge generators $L_a^{\lambda_1 \dots \lambda_s}$, $s = 1, 2, \dots$ of the extended algebra one can add longitudinal terms. All representations are therefore defined modulo longitudinal terms.

³Note that the square mass operator P^2 , is a Casimir invariant for the above algebra while the spin operator $W^\mu W_\mu$ (W^μ being the Pauli-Lubansky vector) is not.

The second general property of the extended algebra is that each gauge generator $L_a^{\lambda_1 \dots \lambda_s}$ cannot be realized as an irreducible representation of the Poincaré subalgebra of a definite helicity, i.e. to be a symmetric and *traceless tensor*. The reason for this is that the commutator of two symmetric traceless generators in the current subalgebra (2) is not any more a traceless tensor. Therefore the gauge generators should realize a reducible representation of the Poincaré subalgebra and each of them carries a sequence of helicities, which we shall find out in the subsequent sections.

Finally, the extended algebra $L_G(\mathcal{SP})$ has a general reducible representation in terms of differential operators of the following form:

$$\begin{aligned} P^\mu &= k^\mu, \\ M^{\mu\nu} &= i(k^\mu \frac{\partial}{\partial k_\nu} - k^\nu \frac{\partial}{\partial k_\mu}) + i(\xi^\mu \frac{\partial}{\partial \xi_\nu} - \xi^\nu \frac{\partial}{\partial \xi_\mu}) - \frac{i}{2} \bar{\vartheta} \gamma^{\mu\nu} \frac{\partial}{\partial \vartheta}, \\ Q_\alpha &= -i \frac{\partial}{\partial \vartheta_\alpha} + i(\gamma^\mu \vartheta)_\alpha k_\mu, \\ L_a^{\lambda_1 \dots \lambda_s} &= \xi^{\lambda_1} \dots \xi^{\lambda_s} \otimes L_a, \end{aligned} \tag{13}$$

where the vector superspace of functions is parameterized in terms of momentum coordinates k^μ , translationally invariant vector variables ξ^μ and anticommuting Grassmann variables ϑ_α

$$\Psi(k^\mu, \xi^\mu, \vartheta_\alpha). \tag{14}$$

This representation allows us to further justify the interpretation of the transformation (12) as a gauge transformation and of the generators $L_a^{\lambda_1 \dots \lambda_s}$ as gauge generators if one considers how this transformation acts on the representation (13). Indeed, the transformation (12) induces a transformation for the vector variable ξ^μ of the form

$$\xi^\mu \rightarrow \xi^\mu + k^\mu, \tag{15}$$

reminiscent of a gauge transformation for the photon polarization vector. Furthermore in order to obtain the irreducible representations from (13), we shall follow Wigner's prescription imposing invariant constraints on the vector space of functions defined in (14) of the following form [11, 12]:

$$k^2 = 0, \quad k^\mu \xi_\mu = 0, \quad \xi^2 = -1. \tag{16}$$

These equations have a unique solution

$$\xi^\mu = \xi k^\mu + e_1^\mu \cos \varphi + e_2^\mu \sin \varphi, \tag{17}$$

where $e_1^\mu = (0, 1, 0, 0)$, $e_2^\mu = (0, 0, 1, 0)$ when $k^\mu = k(1, 0, 0, 1)$, thus justifying the interpretation of the vector variable ξ^μ as a polarization vector⁴. The invariant subspace of functions is now reduced to the form $\Psi(k^\mu, \xi, \varphi, \vartheta_\alpha)$, where ξ and φ remain as independent variables.

There are important properties of the above representation (13), (16) and (17) which are worth mentioning:

(i) The gauge transformation (12), (15) cannot trivialize the above representation by nullifying the generators $L_a^{\lambda_1 \dots \lambda_s}$, but what it can do is to change the parameter ξ in front of k^μ in (17) and

⁴In this article we shall consider only massless representations with $k^2 = 0$.

(ii) This representation is transversal in the sense that

$$k_{\lambda_1} L_a^{\lambda_1 \dots \lambda_s} = 0, \quad s = 1, 2, \dots \quad (18)$$

Having in hand this interpretation of the generators $L_a^{\lambda_1 \dots \lambda_s}$ we can divide the vector space of representations into *pure longitudinal* and *transversal* subsets.

4 Longitudinal Representations

Let us consider an irreducible representation of the superPoincaré algebra (6), in which the generators $P^\mu, M^{\mu\nu}, Q_\alpha^i$ realize a matrix representation with maximal helicity h and the L_a realize an irreducible matrix representation of the internal algebra L_G . If one now takes the gauge generators in the trivial form $L_a^{\lambda_1 \dots \lambda_s} = 0$, $s = 1, 2, \dots$ it is easy to check that this set of generators fulfils all commutation relations of the algebra (9) and therefore forms a true representation of the extended algebra $L_G(\mathcal{SP})$. Applying the above theorem to the representation just described we find that it is isomorphic to the representation in which all generators remain in the same matrix form, except that the gauge generators $L_a^{\lambda_1 \dots \lambda_s}$ are now purely longitudinal. Thus we have the following equivalence relation:

$$\begin{array}{ccc} L_{\mathcal{SP}} : & P^\mu, M^{\mu\nu}, Q_\alpha^i & P^\mu, M^{\mu\nu}, Q_\alpha^i \\ & L_a^{\lambda_1 \dots \lambda_s} = 0, & L_a^{||\lambda_1 \dots \lambda_s} = k^{\lambda_1} \dots k^{\lambda_s} \oplus L_a \\ L_G : & L_a & L_a \end{array} \quad \Leftrightarrow \quad (19)$$

where $s = 1, 2, \dots$. It states that representations with trivial generators $L_a^{\lambda_1 \dots \lambda_s} = 0$ and representations with purely longitudinal generators $L_a^{||\lambda_1 \dots \lambda_s} = k^{\lambda_1} \dots k^{\lambda_s} \oplus L_a$ are isomorphic to each other. In other words, pure longitudinal representations factorize into super-Poincaré $L_{\mathcal{SP}}$ and internal L_G algebra multiplets. Or, if one reads this statement from right to left, it says that pure longitudinal representations of $L_a^{\lambda_1 \dots \lambda_s}$ carry no more helicities than the ones carried by the representation of the SuperPoincaré subgroup, since it is equivalent to a trivial representation of $L_a^{\lambda_1 \dots \lambda_s}$, namely $L_a^{\lambda_1 \dots \lambda_s} = 0$ ($s = 1, 2, \dots$).

5 Transversal Representations

As we have seen in the previous section any representation of the extended algebra (9) in which the generators of the superPoincaré subalgebra (6) realize a matrix representation of finite multiplicity is always equivalent to a representation in which the gauge generators are longitudinal and therefore trivial. It seems natural to think that in order to get a nontrivial representation for the gauge generators one should consider infinite-dimensional representations of the superPoincaré subalgebra (6). Such representations have been constructed in the article [14].

The irreducible representation of the extended algebra can be found by the well-known method of induced representations [13, 14]. This method consists of finding a representation of the Wigner's little group L and boosting it up to a representation of the full group. The subgroup L is a group of transformations which leave a fixed momentum, in our case time-like momentum $k^\mu = k(1, 0, 0, 1)$, invariant. The Poincaré generators in L form the Euclidean algebra $E(2)$ (see Appendix for definitions)

$$[h, \pi'] = i\pi'', \quad [h, \pi''] = -i\pi', \quad [\pi', \pi''] = 0.$$

Notice that transformations generated by the gauge $L_a^{\lambda_1 \dots \lambda_s}$ and supercharge Q_α generators leave the manifold of states with fixed momentum invariant, since they all commute with P^μ , therefore all these generators should be included into the little algebra L, so that we have the following generators in L^5 :

$$h, \quad \pi', \quad \pi'', \quad Q_\alpha, \quad \bar{Q}_{\dot{\alpha}}, \quad L_a^{\lambda_1 \dots \lambda_s}. \quad (20)$$

The full set of commutators of the L algebra are presented in the Appendix and have the following form⁶:

$$\begin{aligned} [h, \pi'] &= +i\pi'', & [h, \pi''] &= -i\pi', & [\pi', \pi''] &= 0, \\ [h, \bar{Q}_1] &= -\frac{1}{2}\bar{Q}_1, & [h, Q_1] &= +\frac{1}{2}Q_1, & \{Q_1, \bar{Q}_1\} &= 4k \\ [\pi', Q_1] &= iQ_2, & [\pi'', Q_1] &= -Q_2, & [\pi', \bar{Q}_1] &= i\bar{Q}_2, & [\pi'', \bar{Q}_1] &= \bar{Q}_2. \end{aligned} \quad (21)$$

The supercharges commute with the gauge generators

$$[Q_1, L_a^\lambda] = 0, \quad [\bar{Q}_1, L_a^\lambda] = 0. \quad (22)$$

The commutators between the E(2) and the $L_a^{\lambda_1}$ generators are:

$$\begin{aligned} [h, L_a^0] &= [h, L_a^3] = 0 & [\pi', L_a^0] &= -iL_a^1 & [\pi'', L_a^0] &= -iL_a^2 \\ [h, L_a^1] &= [h, L_a^2] = 0 & [\pi', L_a^3] &= -iL_a^1 & [\pi'', L_a^3] &= -iL_a^2 \\ [h, L_a^1] &= +iL_a^2 & [\pi', L_a^1] &= -i(L_a^0 - L_a^3) & [\pi'', L_a^1] &= 0 \\ [h, L_a^2] &= -iL_a^1 & [\pi', L_a^2] &= 0 & [\pi'', L_a^2] &= -i(L_a^0 - L_a^3) \end{aligned} \quad (23)$$

and the higher rank generators $L_a^{\lambda_1 \dots \lambda_s}$ have similar structure of commutators (see details in the Appendix). The problem reduces to the construction of the unitary irreducible representations of the L algebra.

The representation of the little algebra L can be found by restricting the general representation (13) into the invariant subspace defined by the conditions (16) and solution (17) to be of the form:

$$\Psi(k^\mu, \xi^\nu, \vartheta_\alpha) \delta(k^2) \delta(k \cdot \xi) \delta(e^2 + 1) = \Phi(k^\mu, \varphi, \xi, \vartheta_\alpha). \quad (24)$$

Making use of the chain rule we may reexpress (13) as a differential operator in the new variables, so that the generators of the L algebra reduce to the form

$$\begin{aligned} h &= -i\frac{\partial}{\partial\varphi} - \frac{1}{2}(\vartheta^1\frac{\partial}{\partial\vartheta^1} - \bar{\vartheta}^1\frac{\partial}{\partial\bar{\vartheta}^1} - \vartheta^2\frac{\partial}{\partial\vartheta^2} + \bar{\vartheta}^2\frac{\partial}{\partial\bar{\vartheta}^2}), \\ \pi' &= \rho \cos \varphi - i\vartheta^1\frac{\partial}{\partial\vartheta^2} - i\bar{\vartheta}^1\frac{\partial}{\partial\bar{\vartheta}^2}, \quad \pi'' = \rho \sin \varphi + \vartheta^1\frac{\partial}{\partial\vartheta^2} - \bar{\vartheta}^1\frac{\partial}{\partial\bar{\vartheta}^2}, \quad \rho = -\frac{i}{k}\frac{\partial}{\partial\xi}, \\ Q_1 &= -i\frac{\partial}{\partial\vartheta^1} - 2ik\bar{\vartheta}^1, \quad Q_2 = -i\frac{\partial}{\partial\vartheta^2}, \\ \bar{Q}_1 &= +i\frac{\partial}{\partial\bar{\vartheta}^1} + 2ik\vartheta^1, \quad \bar{Q}_2 = +i\frac{\partial}{\partial\bar{\vartheta}^2} \end{aligned} \quad (25)$$

and taking into account (17) the gauge generators become

$$L_a^{\perp \mu_1 \dots \mu_s} = \prod_{i=1}^s (\xi k^{\mu_i} + e_1^{\mu_i} \cos \varphi + e_2^{\mu_i} \sin \varphi) \oplus L_a. \quad (26)$$

⁵In this section we shall use two component Weyl spinors and only discuss $N = 1$ supersymmetry.

⁶Not all of them are presented here in the main text.

This is a purely transversal representation and, as we have already mentioned in the previous sections, it cannot be trivialized by the transformations (12), (15). It is transversal in the sense that

$$k_{\lambda_1} L_a^{\perp \lambda_1 \dots \lambda_s} = 0. \quad s = 1, 2, \dots \quad (27)$$

What is important to notice is that the commutators between the generators of $E(2)$ and the $L_a^{\perp \lambda_1 \dots \lambda_s}$ generators of the little algebra L are fulfilled only if $\rho \neq 0$. An example of this is the commutator $[\pi', L_a^0] = -iL_a^1$.

Next we are interested in knowing the helicity content of the transversal gauge generators just constructed. The supercharges Q_1, \bar{Q}_1 carry helicities $h = (1/2, -1/2)$, as one can see from the commutators of the helicity operator with supercharges in (21). The Poincaré generators $\pi^\pm = \pi' \pm \pi''$ carry helicities $h = (1, -1)$. The fact that the $L_a^\pm = L_a^1 \pm iL_a^2$ carry helicities $h = (1, -1)$ is seen from the commutators in the first column of (23):

$$[h, L_a^\pm] = \pm L_a^\pm. \quad (28)$$

The rank-2 generators $L_a^{++}, L_a^{+-}, L_a^{--}$ carry helicities $h = (2, 0, -2)$, where

$$L_a^{++} = L_a^{11} + 2iL_a^{12} - L_a^{22}, \quad L_a^{+-} = L_a^{11} + L_a^{22}, \quad L_a^{--} = L_a^{11} - 2iL_a^{12} - L_a^{22},$$

so that $[h, L_a^{\pm\pm}] = \pm 2L_a^{\pm\pm}$, $[h, L_a^{+-}] = 0$ and in general the rank- s ($L_a^{+\dots+}, \dots, L_a^{-\dots-}$) generators carry helicities in the following range:

$$h = (s, s-2, \dots, -s+2, -s), \quad (29)$$

in total $s+1$ states. (Remember that gauge generator $L_a^{\lambda_1 \dots \lambda_s}$ cannot be realized as an irreducible representation of the Poincaré subalgebra of a definite helicity.) This can be seen also from the explicit representation (26):

$$L_a^{\perp \mu_1 \dots \mu_s} = \prod_{n=1}^s (\xi k^{\mu_n} + e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n}) \oplus L_a, \quad (30)$$

where $e_\pm^\mu = (e_1^\mu \mp i e_2^\mu)/2$. The last formula also illustrates the realization of the transformation rule (12). Indeed if we perform the multiplication in (30) and collect terms with a given power of momentum we get the following expression

$$\begin{aligned} L_a^{\perp \mu_1 \dots \mu_s} &= \prod_{n=1}^s (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n}) \oplus L_a + \\ &+ \sum_1 \xi k^{\lambda_1} \prod_{n=1}^{s-1} (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n}) \oplus L_a + \dots + \xi k^{\lambda_1} \dots \xi k^{\lambda_s} \oplus L_a, \end{aligned} \quad (31)$$

where

$$\prod_{n=1}^s (e^{i\varphi} e_+^{\mu_n} + e^{-i\varphi} e_-^{\mu_n}) \oplus L_a \quad (32)$$

is the transversal part of the generator which we describe in terms of ($L_a^{+\dots+}, \dots, L_a^{-\dots-}$). The rest of the terms (corresponding to the terms with indices 0 or 3) are purely longitudinal, transforming under (12), and can be gauged away⁷.

⁷The situation is analogous to the polarization tensor of the graviton $e^{\mu\nu}(k) = e_1^{\mu\nu} + e_2^{\mu\nu} + k^\mu \xi^\nu + k^\nu \xi^\mu$. The first two terms describe transversal polarizations, the last two terms describe the longitudinal part and if one takes $\xi^\mu = e_1^\mu$ then there will be a spin one part, but still representing a pure gauge.

The states of the representation can be constructed by using these operators. The massless irreducible representation of $\mathcal{N} = 1$ supersymmetry comprises the two states with helicities λ and $\lambda - 1/2$:

$$\begin{array}{cc} |\lambda\rangle & \bar{Q}_1 |\lambda\rangle \\ \lambda & \lambda - 1/2, \end{array} \quad (33)$$

where $h|\lambda\rangle = \lambda|\lambda\rangle$ and $Q_1|\lambda\rangle = 0$. Because the operators π^\pm commute with the supercharges (21)⁸ they generate an infinite tower of high helicity states:

$$\begin{array}{ccccccc} \dots & \pi^+ |\lambda\rangle & & |\lambda\rangle & & \pi^- |\lambda\rangle & \dots \\ \dots & \pi^+ \bar{Q}_1 |\lambda\rangle & & \bar{Q}_1 |\lambda\rangle & & \pi^- \bar{Q}_1 |\lambda\rangle & \dots \\ \dots & \lambda + 1 & & \lambda & & \lambda - 1 & \dots \\ \dots & \lambda + 1/2 & & \lambda - 1/2 & & \lambda - 3/2 & \dots \end{array} \quad (34)$$

From the above formulae it follows that the infinite multiplets built up by any integer λ are isomorphic to each other. The same is true for multiplets built up by any half-integer λ . The supersymmetry transforms simultaneously different pairs of states within the large multiplet, the vertical columns in (34). It does not transform nontrivially the whole multiplet, that is the horizontal states in (34). The operators $(L_a^{++++}, \dots, L_a^{----})$ commute with the supercharges (22) and with the π^\pm generators (23) and are similar to the creation and annihilation operators of the Kac-Moody algebra. Therefore the state $|\lambda\rangle$ must also form an irreducible representation of the internal algebra L_G from which color states of high helicity are generated.

We note that these infinite representations are the same as those found in [14], where the continuous spin representations of the superPoincaré group were derived.

6 Generalization of de Sitter and Conformal Groups

We might ask if the extension above can be made for the de Sitter and the conformal groups, too. Consider first the algebras $SO(4, 1)$ or $SO(3, 2)$

$$[J^{AB}, J^{CD}] = i(g^{AD} J^{BC} - g^{AC} J^{BD} + g^{BC} J^{AD} - g^{BD} J^{AC}),$$

where $g^{AB} = (+ - - -)$ or $g^{AB} = (+ - - +)$ and $A, B = 0, 1, \dots, 4$. The Wigner-Inönü contraction $J^{4\mu} = R P^\mu$, $J^{\mu\nu} = M^{\mu\nu}$, where $\mu, \nu = 0, 1, 2, 3$ and $R \rightarrow \infty$, reduces it to the Poincaré algebra. In the previous analysis there were no restrictions on the dimension of space-time when we considered the bosonic part. We can then drop the translation generators and just consider the sets of commutators

$$\begin{aligned} [J^{AB}, J^{CD}] &= i(g^{AD} J^{BC} - g^{AC} J^{BD} + g^{BC} J^{AD} - g^{BD} J^{AC}), \\ [J^{AB}, L_a^{C_1 \dots C_s}] &= i(\eta^{C_1 B} L_a^{AC_2 \dots C_s} - \dots - \eta^{C_s A} L_a^{C_1 \dots C_{s-1} B}), \\ [L_a^{C_1 \dots C_n}, L_b^{C_{n+1} \dots C_s}] &= i f_{abc} L_c^{C_1 \dots C_s} \quad (s = 0, 1, 2, \dots). \end{aligned} \quad (35)$$

This is an obvious generalization to the cases of the (anti)de Sitter groups.

⁸This is because on the state $|\lambda\rangle$ the supercharges Q_2, \bar{Q}_2 are realized trivially $Q_2|\lambda\rangle = \bar{Q}_2|\lambda\rangle = 0$.

Similarly since the $SO(d, 2)$ algebra is isomorphic to the conformal algebra the algebra $L_G(\mathcal{P})$ can be extended to the conformal group as well with the following well known identification:

$$J^{\mu\nu} = M^{\mu\nu}, \quad J^{\mu,d} = \frac{1}{2}(K^\mu - P^\mu), \quad J^{\mu(d+1)} = \frac{1}{2}(K^\mu + P^\mu), \quad J^{(d+1)d} = D, \quad (36)$$

where $g^{AB} = (+ - - - \dots - +)$ and $A, B = (0, \dots, d, d+1)$. Thus we have the algebra $L_G(\mathcal{SO})$ of the form

$$\begin{aligned} \frac{1}{i}[J^{AB}, J^{CD}] &= g^{AD} J^{BC} - g^{AC} J^{BD} + g^{BC} J^{AD} - g^{BD} J^{AC}, \\ \frac{1}{i}[J^{AB}, L_a^{D_1 \dots D_s}] &= \eta^{D_1 B} L_a^{AD_2 \dots D_s} - \dots - \eta^{D_s A} L_a^{D_1 \dots D_{s-1} B}, \\ \frac{1}{i}[L_a^{D_1 \dots D_n}, L_b^{D_{n+1} \dots D_s}] &= f_{abc} L_c^{D_1 \dots D_s} \quad (s = 0, 1, 2, \dots). \end{aligned} \quad (37)$$

We defer to future work the study of generalisations of these algebras to superalgebras.

7 Conclusions

In this paper we have studied infinite-component massless supermultiplets which arise from a new extension of the superPoincaré algebra. We find that they agree with the continuous spin representations of the ordinary superPoincaré algebra. This provides us with a new framework to discuss such representations. There has been a struggle since the advent of String Theory to describe the zero-tension limit of such a theory, which should be a theory with massless particles of all possible spins [15, 16, 17, 18, 19, 20, 21]. It is hence interesting to study methods to generate infinite-component massless supermultiplets. In this paper we have only taken a first step to include half-integer spins in a recently proposed scheme [1]. In future work we will extend this to higher supersymmetries, higher-dimensional algebras and possibly further extension along the lines of this paper.

Acknowledgement

One of us G.S would like to thank CERN Theory Division for hospitality. L.B. wants to thank Edward Witten for an invitation to the Institute of Advanced Study, where part of the work was done. This work was supported in part by the European Commission under the ERC Advanced Grant 226371 and the contract PITN-GA-2009-237920. I. A. was also supported in part by the CNRS grant GRC APIC PICS 3747.

Appendix

The irreducible representations of the extended algebra can be found by the well-known method of induced representations [13, 14]. This method consists of finding a representation of the Wigner's little group L and boosting it up to a representation of the full group. The subgroup L is a group of transformations which leave a fixed momentum, in our case the time-like momentum $k^\mu = k(1, 0, 0, 1)$, invariant. Under the Lorentz

rotations the action of the element $U_\theta = \exp(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$ creates an infinitesimal transformation $k^\mu \rightarrow \omega^\mu_\nu k^\nu + k^\mu$, and $k^\mu = k(1, 0, 0, 1)$ is left invariant provided the parameters obey the relations $\omega_{30} = 0$, $\omega_{10} + \omega_{13} = 0$, $\omega_{20} + \omega_{23} = 0$. Therefore the little subalgebra L contains at least the following generators:

$$h = M_{12}, \quad \pi'/P^0 = M_{10} + M_{13}, \quad \pi''/P^0 = M_{20} + M_{23}.$$

The M_{12} represents the helicity operator h :

$$h = \frac{\vec{P}\vec{S}}{P^0} = \frac{\vec{P}\vec{J}}{P^0} = \frac{P_i\epsilon_{ijk}M_{jk}}{P^0} = M_{12},$$

where $(\vec{J} = \vec{R} \times \vec{P} + \vec{S})$. The super-Poincaré little algebra is:

$$\begin{aligned} [h, \pi'] &= +i\pi'', & [h, \pi''] &= -i\pi', & [\pi', \pi''] &= 0, \\ [h, \bar{Q}_1] &= -\frac{1}{2}\bar{Q}_1, & [h, Q_1] &= +\frac{1}{2}Q_1, & \{Q_1, \bar{Q}_1\} &= 4k, \\ [h, \bar{Q}_2] &= +\frac{1}{2}\bar{Q}_2, & [h, Q_2] &= -\frac{1}{2}Q_2, & \{Q_2, \bar{Q}_2\} &= 0, \\ [\pi', Q_1] &= iQ_2, & [\pi'', Q_1] &= -Q_2, & [\pi', \bar{Q}_1] &= i\bar{Q}_2, & [\pi'', \bar{Q}_1] &= \bar{Q}_2, \\ [\pi', Q_2] &= 0, & [\pi'', Q_2] &= 0, & [\pi', \bar{Q}_2] &= 0, & [\pi'', \bar{Q}_2] &= 0 \end{aligned} \quad (38)$$

and the rest of the anticommutators between the supercharges is $\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0$. The first level commutation relations in (10) are

$$[M^{\mu\nu}, L_a^\lambda] = i(\eta^{\lambda\nu}L_a^\mu - \eta^{\lambda\mu}L_a^\nu), \quad [Q_\alpha, L_a^\lambda] = 0, \quad [\bar{Q}_{\dot{\alpha}}, L_a^\lambda] = 0$$

and, when written in components, the little subalgebra L takes the form:

$$\begin{aligned} [h, L_a^0] &= [h, L_a^3] = 0 & [\pi', L_a^0] &= -iL_a^1 & [\pi'', L_a^0] &= -iL_a^2 \\ [h, L_a^1] &= [h, L_a^2] = 0 & [\pi', L_a^3] &= -iL_a^1 & [\pi'', L_a^3] &= -iL_a^2 \\ [h, L_a^1] &= +iL_a^2 & [\pi', L_a^1] &= -i(L_a^0 - L_a^3) & [\pi'', L_a^1] &= 0 \\ [h, L_a^2] &= -iL_a^1 & [\pi', L_a^2] &= 0 & [\pi'', L_a^2] &= -i(L_a^0 - L_a^3). \end{aligned}$$

The second level commutation relations are

$$[M^{\mu\nu}, L_a^{\lambda_1\lambda_2}] = i(\eta^{\lambda_1\nu}L_a^{\mu\lambda_2} - \eta^{\lambda_1\mu}L_a^{\nu\lambda_2} + \eta^{\lambda_2\nu}L_a^{\mu\lambda_1} - \eta^{\lambda_2\mu}L_a^{\nu\lambda_1}), \quad [Q_\alpha, L_a^{\lambda_1\lambda_2}] = [\bar{Q}_{\dot{\alpha}}, L_a^{\lambda_1\lambda_2}] = 0$$

and in components they have the form:

$$\begin{aligned} [h, L_a^{00}] &= [h, L_a^{03}] = [h, L_a^{33}] = 0 \\ [h, L_a^{01}] &= +iL_a^{02} \\ [h, L_a^{02}] &= -iL_a^{01} \\ [h, L_a^{13}] &= +iL_a^{23} \\ [h, L_a^{23}] &= -iL_a^{13} \\ [h, L_a^{11}] &= +2iL_a^{12} \\ [h, L_a^{22}] &= -2iL_a^{12} \\ [h, L_a^{12}] &= +i(L_a^{22} - L_a^{11}), \end{aligned} \quad (39)$$

and with translation operators π' and π'' :

$$\begin{aligned}
[\pi', L_a^{00}] &= -2iL_a^{01}, & [\pi'', L_a^{00}] &= -2iL_a^{02} \\
[\pi', L_a^{01}] &= -iL_a^{00} + iL_a^{03} - iL_a^{11}, & [\pi'', L_a^{01}] &= -iL_a^{12} \\
[\pi', L_a^{02}] &= -iL_a^{12}, & [\pi'', L_a^{02}] &= -iL_a^{00} + iL_a^{03} - iL_a^{22} \\
[\pi', L_a^{03}] &= -iL_a^{01} - iL_a^{13}, & [\pi'', L_a^{03}] &= -iL_a^{02} - iL_a^{23} \\
[\pi', L_a^{11}] &= -2iL_a^{01} + 2iL_a^{13}, & [\pi'', L_a^{11}] &= 2iL_a^{23} - 2iL_a^{02} \\
[\pi', L_a^{12}] &= iL_a^{23} - iL_a^{02}, & [\pi'', L_a^{12}] &= iL_a^{13} - iL_a^{01} \\
[\pi', L_a^{13}] &= -iL_a^{11} + iL_a^{33} - iL_a^{03}, & [\pi'', L_a^{13}] &= -iL_a^{12} \\
[\pi', L_a^{22}] &= 0, & [\pi'', L_a^{22}] &= 0 \\
[\pi', L_a^{23}] &= -iL_a^{12}, & [\pi'', L_a^{23}] &= -iL_a^{22} + iL_a^{33} - iL_a^{03} \\
[\pi', L_a^{33}] &= -2iL_a^{13}, & [\pi'', L_a^{33}] &= -2iL_a^{23}.
\end{aligned} \tag{40}$$

The current subalgebra in (11) between generators has the following form:

$$\begin{aligned}
[L_a^0, L_b^0] &= if_{abc}L_c^{00}, \quad [L_a^0, L_b^1] = if_{abc}L_c^{01}, \quad [L_a^0, L_b^2] = if_{abc}L_c^{02}, \quad [L_a^0, L_b^3] = if_{abc}L_c^{03} \\
[L_a^1, L_b^1] &= if_{abc}, \quad [L_a^1, L_b^2] = if_{abc}L_c^{12}, \quad [L_a^1, L_b^3] = if_{abc}L_c^{13} \\
[L_a^2, L_b^2] &= if_{abc}L_c^{22}, \quad [L_a^2, L_b^3] = if_{abc}L_c^{23}, \\
[L_a^3, L_b^3] &= if_{abc}L_c^{33}.
\end{aligned} \tag{41}$$

and so on to the higher levels.

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